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## Renormalons and confinement<sup>\*</sup>

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### Abstract

We compute the renormalon ambiguity of the static potential, in the limit of a large number of flavors. An extrapolation of the QED result to QCD implies that the large distance behavior of the quark potential is arbitrary in perturbation theory, as there are an infinite number of prescriptions to assign. The shape of the potential at large distances is not only affected by the renormalon pole closest to the origin of the Borel plane, but a resummation of all renormalon contributions is required. In particular, confinement can be accommodated, but it is not explained. At short distances there is no indication of a linear term in the potential.

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A way to learn about intrinsic limitations of perturbation theory in the determination of a physical quantity is via the study of the large order behavior of the perturbative expansion. Although the coupling constant depends only logarithmically on the scale, the perturbative expansion can produce power suppressed corrections, as it is an asymptotic series. Some nonperturbative features of the theory can be inferred from the positions and residues of the singularities in the Borel transform, with respect to the coupling constant [1].

At present it is not known how to compute renormalons in full QCD. Most calculations have been performed by summing fermion bubble insertions in the gluon propagator. The neglected graphs are presumably just as important as those taken into account, so the result of summing the fermion bubble chain does not prove the existence of renormalons in QCD. Calculating renormalons via bubble summation is rigorous in the limit of QED with a large number of fermions,  $N_f \rightarrow \infty$ . In this case a fermion bubble insertion in a photon line yields a factor of  $a = \alpha N_f$ , which can be summed to all orders. The contribution of other diagrams is suppressed by powers of  $1/N_f$ . Extrapolating the large  $N_f$  QED result to QCD is accomplished by assuming that the ultraviolet and infrared properties can be interchanged. Such studies have received considerable attention in the literature recently, mostly in connection with the heavy quark expansion (see, for example [2–9]).

In this letter we study renormalon ambiguities in a static property of QCD, *i.e.*, the potential between a heavy quark and an anti-quark. Neglecting color factors, the leading order one gluon exchange diagram contributes to the potential by the usual Coulomb term

$$V_0(r) = -16\pi^2 \alpha \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} = -\frac{g^2}{r}. \quad (1)$$

Diagrams with multiple gluon exchange are suppressed by powers of  $1/N_f$ , while the insertion of fermion bubbles in the gluon line involves the coupling  $a$ , which is treated as a parameter of order unity. Therefore, we have to consider an arbitrary number of bubbles inserted into the one gluon exchange diagram.

If the perturbative expansion of the potential is given by

$$V(r) = \sum_{n=0}^{\infty} V_n (b_0 \alpha)^{n+1}, \quad (2)$$

then the Borel transform is defined as

$$\tilde{V}(r) = \sum_{n=0}^{\infty} \frac{V_n}{n!} u^n. \quad (3)$$

Here  $u$  is the Borel parameter. For convenience, we defined the Borel transformation with respect to  $b_0 \alpha$ , where  $b_0$  is the first coefficient of the  $\beta$ -function,  $\beta(\alpha) = \mu^2 \partial \alpha / \partial \mu^2 = -b_0 \alpha^2 + \mathcal{O}(\alpha^3)$ . In QED  $b_0 = -N_f/3\pi$ , while in QCD  $b_0 = (11 - \frac{2}{3}N_f)/4\pi$  [10]. The reason for studying the Borel transform  $\tilde{V}(r)$  is that it may converge even if the perturbative series for  $V(r)$  in Eq. (2) is divergent. In that case the inverse Borel transformation

$$V(r) = \int_0^\infty du e^{-u/(b_0 \alpha)} \tilde{V}(r), \quad (4)$$

provides a good definition of  $V(r)$ . However, if  $V_n$  grows at least factorially for large  $n$ , then  $\tilde{V}(r)$  can have singularities on the integration contour. In our case these singularities will be isolated poles on the positive real axis that arise from the infrared region of Feynman diagrams. These so-called infrared renormalons yield ambiguities in the reconstruction of  $V(r)$  from  $\tilde{V}(r)$ , as the result depends on the regularization of the pole contributions.

The Borel transform of the static potential can be calculated directly, using the Borel transform of the resummed gluon propagator [11]. This yields

$$\begin{aligned} \tilde{V}(r) &= -\frac{16\pi^2}{b_0} \left( \frac{e^C}{\mu^2} \right)^{-u} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{r}}}{(\vec{k}^2)^{1+u}} \\ &= -\frac{4 e^{-Cu}}{b_0} \frac{1}{r} (\mu r)^{2u} \frac{\Gamma(\frac{1}{2} + u) \Gamma(\frac{1}{2} - u)}{\Gamma(2u + 1)}, \end{aligned} \quad (5)$$

where  $\mu$  is the renormalization scale, and  $C$  is a regularization scheme dependent constant. We work in the momentum subtraction scheme in which  $C = 0$ . The singularities of  $\tilde{V}(r)$  are simple poles along the positive real axis at half-integer values of  $u$

$$u_k = k + \frac{1}{2}, \quad k = 0, 1, 2, \dots \quad (6)$$

The residues of these poles are

$$\text{Res} \left[ \tilde{V}(r) \Big|_{u_k} \right] = \frac{1}{r} \frac{4}{b_0} (\mu r)^{2k+1} \frac{(-1)^k}{(2k+1)!}. \quad (7)$$

The physical potential is reconstructed by the inverse Borel transform in Eq. (4)

$$V(r) = -\frac{1}{r} \frac{4}{b_0} \int_0^\infty du (\Lambda r)^{2u} \frac{\Gamma(\frac{1}{2} + u) \Gamma(\frac{1}{2} - u)}{\Gamma(2u + 1)}, \quad (8)$$

where  $\Lambda$  is the pole of the coupling (the dynamically generated scale), and we have used the relation  $\Lambda^2/\mu^2 = e^{-1/(b_0 \alpha[\mu^2])}$ . To carry out this integration, the contour has to be deformed away from the renormalon poles.\* Using the QCD  $\beta$ -function coefficient, we obtain the ambiguity in the quark potential

$$\Delta V(r) = \frac{1}{r} \frac{4}{b_0} \sum_{k=0}^\infty c_k (\Lambda r)^{2k+1} \frac{(-1)^k}{(2k+1)!}. \quad (9)$$

Here  $c_k$  are arbitrary complex numbers of order unity, related to the ambiguity of regularizing the  $1/(u - u_k)$  poles of  $\tilde{V}(r)$  by their principal value plus  $c_k$  times the delta function  $\delta(u - u_k)$ .

Inserting the QCD  $\beta$ -function into Eq. (8) is equivalent to replacing the coupling constant evaluated at the fixed subtraction scale in the one gluon exchange diagram with the running coupling at the scale of the gluon momentum,  $\alpha(k^2)$ . Diagrammatically this corresponds to dressing the one gluon exchange diagram with self-energy and vertex corrections computed in the leading logarithmic approximation.

The crucial point is that the static potential is an unambiguous physical quantity. It can be calculated on the lattice as the expectation value of a Wilson loop. In view of the above ambiguity originating from the renormalon calculation, the static potential can only be free of ambiguities if nonperturbative effects cancel Eq. (9). It is in this sense how renormalons ‘trace’ nonperturbative effects.†

There are a number of points to be made regarding our main result in Eq. (9):

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\*These renormalon poles are not present if we include an infrared regulator, like a gluon mass or a vacuum expectation value for the  $\langle G_{\mu\nu} G^{\mu\nu} \rangle$  operator. However, we are interested in what can be learned strictly from perturbation theory.

†There may also be instanton contributions, and nonperturbative effects that do not correspond to any local operator.

*a.* In previous calculations, powers of some large mass scale or momentum flow suppressed the renormalon contributions of the poles further away from the origin of the Borel plane. In our case this scale is replaced by  $1/r$ . At large distances  $\Lambda/(1/r)$  enhances the contributions of the poles far from the origin. Therefore, the ambiguity in the static potential at large distances is affected by all renormalon poles.

*b.* By varying the values of  $c_k$ , the renormalon ambiguity at large distances becomes arbitrary. Taking for example  $c_k = 1$  yields

$$\Delta V(r) \sim \frac{\sin(\Lambda r)}{r}, \quad (10)$$

while  $c_k = (-1)^k$  yields

$$\Delta V(r) \sim \frac{\sinh(\Lambda r)}{r}. \quad (11)$$

In the first case, the renormalon calculation implies that nonperturbative correction to the potential is not confining, while it is confining in the latter case. With a proper choice of the  $c_k$ 's, an arbitrary potential shape can be produced at large distances. For example, the  $c_k$ 's can reproduce a nonperturbative correction to the potential of the form  $\Lambda \sqrt{1 + (\Lambda r)^2}$ , *i.e.*, a linear behavior in  $r$  for  $r \gg \Lambda^{-1}$ .

*c.* The renormalon ambiguities do not affect the Coulomb part of the potential, only terms that are proportional to  $r^{2k}$ . Near  $r = 0$  the renormalon calculation gives no indication of a linearly rising correction to the potential. It seems conceivable to us that this is a real feature of QCD, namely 'flux tubes' develop only for  $r > \Lambda^{-1}$ . Our result does not constrain the large  $r$  behavior: as was shown above,  $V(r)$  can even grow linearly for large  $r$ .

*d.* There is an analogy with non-renormalizable theories, where knowledge of an infinite number of counterterms is needed to derive scattering amplitudes from the Lagrangian. In the present case, the values of  $c_k$  would specify the gluon propagator at large distances.

These observations imply that the renormalon calculation supplemented with bubble summation has a very limited capability in describing long distance QCD. In this sense our results question the suitability of resumming fermion bubble chains for phenomenological

estimates of nonperturbative effects in real QCD. One could argue that some values of  $c_k$  imply confinement, but based on the information available from the renormalon calculation, such a choice is completely arbitrary, and is not any better motivated than values of  $c_k$  that suggest that nonperturbative contributions to the static potential need not be confining. In this framework, therefore, the Landau pole gives no indication of what type of nonperturbative effects occur. Beyond the perturbatively calculable result there are an infinite number of parameters that need to be specified to describe the long distance dynamics.

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